

ON RIGID SINGULARITIES*

by Michael Schlessinger

A local complex space X is *rigid* if every flat family $(X_t, t \in T)$ which contains X is locally trivial, in the sense that the total space $V = \cup X_t$ is locally a product $X \times T$. Equivalently, every sufficiently near deformation X_t of X is isomorphic to X (at least when T is normal). Singularities X of this type must therefore be "generic" members of any families which contain them, for no other singularity can specialize to X .

This paper exhibits several classes of rigid singularities, many of which have already appeared in the literature in one form or another, and, principally, an extensive class of singularities which cannot be deformed to a rigid singularity. The examples of these last given here are conical singularities, cones over projective manifolds, every deformation of which is still conical. However, the singularities in these examples tend to be generic, in the sense that they cannot be realized as the specialization of other singularities.

In a related paper, pp. 113–117 in this volume, David Mumford extends and sharpens these results in the low-dimensional case not treated here.

In outline, §1 is a brief survey of the cotangent spaces used to construct the versal deformation of a singularity (Grauert, [4]). §2 contains the basic comparison theorem between the deformations of a local complex space and the complement of its origin, upon which most of the examples of rigid singularities in §3 are based. §4 compares the deformations of the cone over a projective manifold Y to the projective deformations of Y , and §5 exhibits the "generic," but non-rigid singularities.

We deal throughout with local complex spaces X , by which is meant the germ of a complex analytic space, not necessarily reduced, at a point, always taken to be origin in some \mathbb{C}^n . Thus X is represented by a complex analytic subspace of some convenient neighborhood of the origin, and \mathcal{O}_X denotes the structure sheaf of X on this neighborhood.

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§1. Infinitesimal deformations and obstructions

1.1 Deformations

Let $X \subset \mathbb{C}^n$ be a local complex space. A deformation of X is a flat map $\pi: V \rightarrow T$ of local complex spaces, together with an isomorphism $\pi^{-1}(0) \cong X$. We say that X may be deformed into the nearby fiber $\pi^{-1}(t)$, for $t \in T$.

If X is defined in some open neighborhood of the origin in \mathbb{C}^n by holomorphic equations $f_i(x) = 0$ ($1 \leq i \leq m$), then in $\mathbb{C}^n \times T$, V has equations $f_i(x, t) = 0$, with $f_i(x, 0) = f_i(x)$. It follows from Grothendieck's fundamental criterion for flatness [2] that V is flat over T if and only if every relation $r(x) = (r_1(x), \dots, r_m(x))$ between $f_1(x), \dots, f_m(x)$ ($\sum r_i(x)f_i(x) = 0$) extends to a relation $r(x, t)$ between the $f_i(x, t)$. (And indeed, we may take this condition as a working definition of flatness.) When X is smooth, π is flat if and only if it is smooth (maximal rank).

Two deformations $\pi_1: V_1 \rightarrow T$, $\pi_2: V_2 \rightarrow T$, are isomorphic if there is a map $\sigma: V_1 \rightarrow V_2$ over T , respecting the isomorphisms

$$\pi_1^{-1}(0) \cong X \cong \pi_2^{-1}(0).$$

σ is then necessarily an isomorphism, by the flatness of π_1 and π_2 .

1.2 T_X^1

Let $I = (f_1, \dots, f_m)$ be the ideal in $\mathcal{O}_{\mathbb{C}^n}$ defining the local complex space $X \subset \mathbb{C}^n$. Thus $\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^n}/I$.

If we take as parameter space the (one point) space $T = \text{Spec } \mathbb{C}\{\varepsilon\}/(\varepsilon^2)$, then a deformation $V \rightarrow T$ of X is a first-order infinitesimal deformation of X . V will be given by equations

$$f_i(x) + \varepsilon g_i(x) = 0$$

and the condition for flatness is simply that the g_i determine an element of the normal sheaf

$$N_X = \text{Hom}_{\mathcal{O}_{\mathbb{C}^n}}(I, \mathcal{O}_X).$$

The ensuing deformation $V \rightarrow T$ is trivial (isomorphic to $X \times T \rightarrow T$) if and only if there is an automorphism $x_j \mapsto x_j + \varepsilon \delta_j(x)$ of $\mathbb{C}^n \times T$ over T such that $(f_i(x + \varepsilon \delta(x)))$ and $(f_i + \varepsilon g_i)$ determine the same ideal in $\mathcal{O}_{\mathbb{C}^n \times T}$; in other words

$$\sum_j \frac{\partial f_i}{\partial x_j} \delta_j(x) = g_i(x) \pmod{I}.$$

Now there is a homomorphism $\rho: \Theta_{\mathbb{C}^n} \rightarrow N_X$ (Θ denoting tangent sheaf) defined by mapping the vector field $\theta = \sum_j \delta_j(x) \partial / \partial x_j$ to the homomorphism

$f_i \mapsto \theta(f_i)$ in N_X , and the element g in N_X induces a trivial deformation of X if and only if it lies in the image of ρ .

Let us define an \mathcal{O}_X -module T_X^1 by the exact sequence

$$(1) \quad 0 \rightarrow \Theta_X \rightarrow \Theta_{\mathbb{C}^n}|_X \rightarrow N_X \rightarrow T_X^1 \rightarrow 0.$$

Then T_X^1 is the set of *isomorphism classes* of first order infinitesimal deformations of X , analogous to $H^1(Y, \Theta_Y)$ for a manifold Y . Notice that T_X^1 is supported on the singular locus of X , so that when X has an isolated singularity at its origin, T_X^1 is a vector space of *finite dimension*.

The module T_X^1 is independent of the choice of the embedding $X \subset \mathbb{C}^n$, up to canonical isomorphism. $T_X^1 = 0$ if X is smooth.

1.3 T_X^2

Consider the problem of extending a deformation $V \rightarrow T$ of X to an infinitesimally larger space $T' \supset T$. Thus, T is defined by an ideal J in \mathcal{O}_T , such that the product $m_T \cdot J$ is 0 (m_T denoting the maximal ideal of \mathcal{O}_T). For simplicity we take the case where T is the n th infinitesimal neighborhood of the origin in \mathbb{C}^1 , defined by $t^n = 0$, and T' is defined by $t^{n+1} = 0$; J is then the one-dimensional vector space $t^n \cdot \mathbb{C}$.

If $f_i(x, t) \pmod{t^n}$ are equations for V and $r(x, t)$ a typical relation, then we have the dot product

$$r(x, t) \cdot f(x, t) = \sum_{i=1}^m r_i(x, t) f_i(x, t) \equiv 0 \pmod{t^n}.$$

In order to lift $V \rightarrow T$ to T' , we seek m -tuples Δf and $\Delta r, \equiv 0 \pmod{t^n}$, such that the dot product $(r + \Delta r) \cdot (f + \Delta f)$ is $\equiv 0 \pmod{t^{n+1}}$ for each r .

If we take a presentation

$$0 \longrightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} I \longrightarrow 0$$

of I as an $\mathcal{O}_{\mathbb{C}^n}$ -module (F free over $\mathcal{O}_{\mathbb{C}^n}$) then

$$r \rightarrow r \cdot f$$

defines an *obstruction* homomorphism from R to $\mathcal{O}_X \otimes_{\mathbb{C}} J$. V extends to $V' \rightarrow T'$ if and only if this obstruction homomorphism is induced by a homomorphism $\delta: F \rightarrow \mathcal{O}_X \otimes J$. The obstruction homomorphism will vanish on "trivial" relations of the form

$$\alpha(a)b - \alpha(b)a$$

(a, b in F). If R_0 denotes the submodule of trivial relations, then we define the \mathcal{O}_X -module T_X^2 by the exact sequence

$$\mathrm{Hom}(F, \mathcal{O}_X) \rightarrow \mathrm{Hom}(R/R_0, \mathcal{O}_X) \rightarrow T_X^2 \rightarrow 0.$$

Then, the *infinitesimal obstructions* to extending deformations of X lie in $T_X^2 \otimes_{\mathbb{C}} J$ for various vector spaces J . Notice that $R_0 = R$ if and only if X is a so-called local complete intersection in \mathbb{C}^n (i.e., I is generated by a regular sequence f_1, \dots, f_m , $m = n - \dim X$), so that $T_X^2 = 0$ when X is a local complete intersection. T_X^2 is, up to a canonical isomorphism, independent of the embedding $X \subset \mathbb{C}^n$ chosen, and is supported on the non-smooth locus of X (in fact, on the non-complete intersection locus of X). Thus T_X^2 is a vector space of finite dimension if X has isolated singularity.

In contrast, the space $T_X^0 = H^0(X, \Theta_X)$ of infinitesimal automorphisms of X has infinite dimension as a vector space, unless X has dimension 0.

Remark. We observe that if Ω denotes Kähler differentials there are inclusions

$$\begin{aligned} \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_X, \mathcal{O}_X) &\hookrightarrow T_X^1 \\ \mathrm{Ext}_{\mathcal{O}_X}^1(I/I^2, \mathcal{O}_X) &\hookrightarrow T_X^2 \\ \mathrm{Ext}_{\mathcal{O}_X}^2(\Omega_X, \mathcal{O}_X) &\hookrightarrow T_X^2, \end{aligned}$$

The first two are isomorphisms when X has *positive depth* (§2) along its singular locus (e.g., when X is reduced of positive dimension), the third is an isomorphism when this depth is > 1 (e.g., X normal) [9]. As Tijourina shows [14], the T_X^i may be replaced by $\mathrm{Ext}^i(\Omega_X, \mathcal{O}_X)$ in the above discussion when X has positive depth along the singular locus.

The cotangent spaces T_X^i are discussed in detail in [9].

1.4 The Versal Deformation

Let X be a local complex space with isolated singularity. In [3] Grauert constructs a *versal deformation* $V \rightarrow S$ of X from which every other deformation $W \rightarrow T$ may be deduced, up to isomorphism, by a map $\phi: T \rightarrow S$, with $\phi^*(V) \cong W$. Moreover, the map

$$l_\phi: l_T \rightarrow l_S$$

between Zariski tangent spaces is uniquely determined by the isomorphism class of W (though ϕ itself is not, in general). As Grauert shows, the Zariski tangent space of S is isomorphic to T_X^1 .

X is *rigid* when every deformation is trivial, or S is reduced to a point. Thus, $T_X^1 = 0$ is the necessary and sufficient condition for rigidity.

By the Artin Approximation Theorem [1] and the existence of a formal versal deformation [11], this condition applies even when X has a non-

isolated singularity; also, a slight modification of Grauert's argument in §3, part 3 of [4] shows that the versal deformation exists whenever T_X^1 has finite dimension.

Notice that a smooth space is trivially rigid.

Example. Let $X \subset \mathbb{C}^n$ be a hypersurface $f(x_1, \dots, x_n) = 0$ with isolated singularity at the origin. One computes easily from 1.2 that

$$T_X^1 \cong \mathcal{O}_{\mathbb{C}^n} / \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Let $g_1(x), \dots, g_r(x)$ be holomorphic functions in $\mathcal{O}_{\mathbb{C}^n}$ which induce a basis of T_X^1 , $S = \mathbb{C}^r(t)$, and define

$$V: f(x) + \sum_{i=1}^r t_i g_i(x) = 0.$$

Then the projection $\pi: V \rightarrow S$ gives the versal deformation of X . Clearly, X is rigid ($T_X^1 = 0$) if and only if some partial $\partial f / \partial x_i(0) \neq 0$, so that a singular hypersurface is never rigid. In fact, if we pick $g_1 \equiv 1$, then X may be deformed into $f + t_1 = 0$, which, for $t_1 \neq 0$, is smooth; so for generic $t \in S$, $\pi^{-1}(t)$ is smooth.

Entirely similar remarks apply to complete intersections.

§2. A Comparison Theorem

If X is not a local complete intersection, the direct computation of T_X^1 is awkward. However, it is frequently possible to compare the deformation theory of X with that of a suitable smooth (non local) variety, where more may be known. For example, if $Z \subset X$ is the singular locus of X , then $U = X - Z$ is smooth, and one can transfer questions on X to ones on U if the depth is large enough. Our definition here is: for a coherent sheaf F on X , $\text{depth}_Z F \geq d$ if and only if the local cohomology $H_Z^p(X, F)$ vanishes for all $p < d$ [7]. Equivalently there is a regular F -sequence f_1, \dots, f_d with $f_i = 0$ on Z . We write $\text{depth}_Z X$ for $\text{depth}_Z(\mathcal{O}_X)$.

Notice, by the long exact sequence

$$0 \rightarrow H_Z^0(X, F) \rightarrow H^0(X, F) \rightarrow H^0(U, F) \rightarrow H_Z^1(X, F) \rightarrow \dots$$

and the vanishing of $H^i(X, F)$ for $i > 0$ (X is Stein), that the condition $\text{depth}_Z F \geq 2$ is equivalent to

$$H^0(X, F) \xrightarrow{\sim} H^0(U, F)$$

and that $H^{p-1}(U, F) \cong H_Z^p(X, F)$ for $p \geq 2$.

If X is smooth, then $\text{depth}_Z X$ equals the codimension of Z in X .

As a simple generalization of Theorem 2 in [12], we have

Theorem 1. *Let X be a local complex space defined in an open neighborhood A of the origin in \mathbf{C}^n , $Z \subset X$ a closed complex subspace containing the singular locus of X , and put $U = X - Z$. If $\text{depth}_Z X \geq 2$, there are exact sequences*

$$(1) \quad 0 \rightarrow T_X^1 \rightarrow H^1(U, \Theta_U) \rightarrow H^1(U, \Theta_{\mathbf{C}^n}|U)$$

$$(1') \quad H^0(U, \Theta_{\mathbf{C}^n}|U) \rightarrow H^0(U, N_U) \rightarrow T_X^1 \rightarrow 0$$

and an inclusion

$$(2) \quad T_X^2 \hookrightarrow H^1(U, N_U)$$

where N_U is the normal bundle of U in \mathbf{A} , and Θ denotes the tangent bundle.

If $\text{depth}_Z X \geq 3$, then

$$(3) \quad T_X^1 \cong H^1(U, \Theta_U)$$

$$(4) \quad T_X^2 \cong H^1(U, N_U).$$

Proof. Let I be the ideal sheaf of X in \mathbf{C}^n . By the definition of T_X^1 in 1.2 we have

$$(5) \quad 0 \rightarrow H^0(X, \Theta_X) \rightarrow H^0(X, \Theta_{\mathbf{C}^n}|X) \rightarrow H^0(X, N_X) \rightarrow T_X^1 \rightarrow 0.$$

In this sequence the first three sheaves are reflexive, being the duals, respectively, of Ω_X , $\Omega_{\mathbf{C}^n}|X$, I/I^2 , and thus as in §1, Lemma 1, of [12], the depth of each is also ≥ 2 . In (5) we can therefore replace $H^0(X, \cdot)$ by $H^0(U, \cdot)$. As $T_X^1|U = 0$ (U is smooth), the sequence

$$(6) \quad 0 \rightarrow \Theta_U \rightarrow \Theta_{\mathbf{C}^n}|U \rightarrow N_U \rightarrow 0$$

is exact, and its cohomology sequence yields (1) and (1').

In 1.3 we have observed that $T_X^2 \cong \text{Ext}^1(I/I^2, \mathcal{O}_X)$. Taking a presentation $0 \rightarrow R \rightarrow F \rightarrow I/I^2 \rightarrow 0$, where F is free over \mathcal{O}_X , and applying $\text{Hom}(\cdot, \mathcal{O}_X)$ we have

$$0 \rightarrow N_X \rightarrow F^* \rightarrow R^* \rightarrow T_X^2 \rightarrow 0,$$

where N_X , F^* , and R^* are the respective duals. Again these sheaves have $\text{depth}_Z \geq 2$, and $T_X^2|U = 0$, so that we can apply $H^0(U, \cdot)$ to obtain (2).

(3) and (4) follow immediately from (1) and (2), since $\Theta_{\mathbf{C}^n}|U$ and F are free and $H^1(U, \mathcal{O}_X) \cong H_Z^1(X, \mathcal{O}_X) = 0$ if $\text{depth}_Z X \geq 3$. Q.E.D.

This theorem compares the infinitesimal deformations and obstructions for X and U . We can conclude that (formal) versal parameter spaces of each are isomorphic, by the lemma below. We first define in general the tangent and obstruction spaces for a local complex space S .

Definition. Let $S \subset \mathbb{C}^n$ be a local complex space, with minimal embedding dimension n . Thus we have

$$\mathcal{O}_S = \mathcal{O}_{\mathbb{C}^n} / J$$

where $J \subset \mathfrak{m}^2$, \mathfrak{m} being the maximal ideal in $\mathcal{O}_{\mathbb{C}^n}$. We define the tangent and obstruction spaces of S :

$$t_S = (\mathfrak{m}/\mathfrak{m}^2)^*$$

$$O_S = (J/\mathfrak{m} \cdot J)^*$$

where $*$ denotes dual vector space.

These definitions are independent of the embedding $S \subset \mathbb{C}^n$; in fact t_S is $T^0(S, \mathbb{C})$ and O_S is $T^1(S, \mathbb{C})$ [9]. Notice that t_S has dimension n , the embedding dimension, and $O_S = 0$ if and only if S is smooth.

If $V \rightarrow S$ is the versal deformation of a local complex space, then one can show that there are canonical maps

$$t_S \xrightarrow{\sim} T_X^1$$

$$O_S \hookrightarrow T_X^2.$$

If $V' \rightarrow S'$ is the versal (formal) deformation of the smooth space U , then there are canonical maps

$$t_{S'} \xrightarrow{\sim} H^1(U, \Theta)$$

$$O_{S'} \hookrightarrow H^2(U, \Theta).$$

Proposition. Let $\phi: S \rightarrow S'$ be a map of local complex spaces. If ϕ induces an isomorphism

$$t_S \xrightarrow{\sim} t_{S'}$$

and an inclusion

$$O_S \hookrightarrow O_{S'}$$

then ϕ is an isomorphism.

Conversely, if ϕ is an isomorphism then both of the above maps are isomorphisms.

Proof. If $t_S \simeq t_{S'}$, then we may take

$$\mathcal{O}_S \cong \mathcal{O}_{C^n}/J$$

$$\mathcal{O}_{S'} \cong \mathcal{O}_{C^n}/J'$$

with both ideals included in the square of the maximal ideal \mathfrak{m} . Thus, ϕ is represented by an automorphism θ of \mathcal{O}_{C^n} such that $\theta(J') \subset J$. If, as we assume, the induced map $J'/\mathfrak{m}J' \rightarrow J/\mathfrak{m}J$ is surjective, then by the Nakayama lemma, θ is a surjection, so that $\theta(J') = J$ and ϕ is an isomorphism.

The converse is trivial.

It is not clear that $U = X - Z$ has a versal deformation. But as long as $H^1(U, \Theta_U)$ has finite dimension. U has a formal versal deformation $V' \rightarrow S'$ (S' being the spectrum of some complete local ring). One can pass from formal deformations of X to formal deformations of U by subtracting Z , and from (6) we find $H^1(U, N_U) \simeq H^2(U, \Theta_U)$ if $\text{depth}_Z X \geq 3$. Thus the formal parameter spaces S and S' of X and U are isomorphic if $\text{depth}_Z X \geq 3$, and U is smooth.

§3. Examples of Rigid Singularities

Most of these are based on the comparison

$$(1) \quad 0 \rightarrow T_X^1 \rightarrow H^1(U, \Theta_U) \rightarrow H^1(U, \Theta_{C^n}|U)$$

of §2, valid if $U = X - Z$ is smooth and $\text{depth}_Z X \geq 2$. Recall that the term $H^1(U, \Theta_{C^n}|U) = 0$ when $\text{depth}_Z X \geq 3$.

a) Fans (Rim)

These are singularities obtained by joining together smooth spaces X_i along a common smooth subspace Z of codimension ≥ 2 in each X_i . Supposing that each X_i is defined in a common neighborhood of the origin in C^n , $Z = X_i \cap X_j$ ($i \neq j$), then we have $X = \cup X_i$ and

$$\mathcal{O}_X = \mathcal{O}_{X_1} \times_{\mathcal{O}_Z} \cdots \times_{\mathcal{O}_Z} \mathcal{O}_{X_m}.$$

That is, \mathcal{O}_X is the set of (f_1, \dots, f_m) with a common image in \mathcal{O}_Z .

Here the depth of X along Z is 1, but Rim's computations with generators and relations [10] show that X is rigid.

The method here yields examples of (non-rigid) non-isolated singularities X with *finite codimension*, in the sense that $\dim T_X^1$ is finite. We can take $X = \cup X_t$, $t \in T$, where X_t , ($t \neq 0$) is a rigid singularity approaching X_0 , not rigid. For example, let $\pi: V \rightarrow T$ be a deformation of the singularity Y , $\text{depth } Y \geq 2$, with $\pi^{-1}(t)$ smooth, $t \neq 0$, and suppose that $\sigma: T \rightarrow V$ is a

section. If we let X be the union $V \cup \mathbb{C}^2$, joined along $\sigma(T)$ and 0, T_X^1 is concentrated at the origin of X , hence has finite dimension.

b. Quotient Singularities

Let a finite group G act by holomorphic automorphisms of \mathbb{C}^m , leaving the origin fixed. (The action is then linear, in suitable coordinates.) If G acts *freely* outside some G invariant complex subspace W (through the origin) of codimension ≥ 3 , then $X = \mathbb{C}^m/G$ is rigid.

The proof is the same as the one given in [12] for the case $W = \{0\}$. Let $V = \mathbb{C}^m - W$, $U = X - Z = V/G$. From $\mathcal{O}_X = (\mathcal{O}_{\mathbb{C}^m})^G$, $H^0(U, \mathcal{O}_U) = H^0(V, \mathcal{O}_V)^G$, where the superscript G denotes the subspace of G invariant elements, we see that $\text{depth}_Z X = \text{depth}_W \mathbb{C}^m \geq 3$. Then

$$T_X^1 \cong H^1(U, \Theta_U) \cong H^1(V, \Theta_V)^G \cong (T_{\mathbb{C}^m}^1)^G = 0.$$

I am indebted to Van de Ven for pointing out that [12] could be so generalized. Again, we may replace \mathbb{C}^m by a rigid space Y , smooth outside of W , for which $\text{depth}_W \mathcal{O}_Y \geq 3$.

This example certainly fails in codimension 2: all of the rational double points are quotient singularities [3], and they are all hypersurfaces in \mathbb{C}^3 .

If Y is not rigid, then we still get

$$T_X^i \cong (T_Y^i)^G$$

under the above assumptions.

This method yields non conical (i.e., non homogeneous) rigid singularities. For example if $\mathbb{Z}/3$ acts on \mathbb{C}^3 by

$$(s, t, u) \rightarrow (\zeta s, \zeta^{-1} t, \zeta u), \quad \zeta = e^{2\pi i/3},$$

we have invariants $st, tu, s^2u, su^2, s^3, t^3, u^3$. In \mathcal{O}_X we have therefore a relation

$$s^3 \cdot t^3 \cdot u^3 = (s^2u)(tu)^2st,$$

so that $s^3 \cdot t^3 \cdot u^3$ is in the fourth power of the maximal ideal of \mathcal{O}_X . As the classes of s^3, t^3, u^3 are not zero in the graded ring of \mathcal{O}_X and their product is 0, this graded ring cannot be equal to \mathcal{O}_X , which is a domain.

c. Cones

The cone over a strongly rigid projective manifold is rigid. Here, a projective manifold $Y \subset \mathbb{P}^n$, $\dim Y > 0$, is *strongly rigid* if

- (i) Y is projectively normal
- (ii) $H^1(Y, \Theta_Y(v)) = 0, \quad -\infty < v < \infty$
- (iii) $H^1(Y, \mathcal{O}_Y(v)) = 0, \quad -\infty < v < \infty.$

Projective normality means simply that the cone $C_Y \subset \mathbb{C}^{n+1}$ (the set of zeroes of the homogeneous polynomials defining Y) is normal at its vertex, O . $F(v)$ denotes the sheaf F tensored with the v th power of the sheaf of the hyperplane bundle.

The deformation of cones is discussed in detail in §4. Briefly, if U is the complement of the vertex in C_Y , then we have a long exact sequence

$$\cdots \rightarrow H^1(U, \mathcal{O}_U) \rightarrow H^1(U, \Theta_U) \rightarrow H^1(U, p^* \Theta_Y) \rightarrow H^2(U, \mathcal{O}_U) \rightarrow \cdots$$

in which p is the projection $U \rightarrow Y$ and the first and third terms are isomorphic respectively to $\sum H^1(Y, \mathcal{O}_Y(v))$ and $\sum H^1(Y, \Theta_Y(v))$ summed over $-\infty < v < \infty$.

This example applies to $Y = \mathbb{P}^a \times \mathbb{P}^b$ in its Segre embedding ($a + b \geq 3$), due originally to Thom and Grauert-Kerner [5], and to $Y = \mathbb{P}^m$ ($m \geq 2$) in any of its Veronese embeddings by hypersurfaces of degree d (also a quotient singularity). If $\dim Y \geq 2$ and Y is rigid and regular, then a sufficiently high Veronese embedding of Y will be strongly rigid.

d. Total deformation space

Let $V \rightarrow S$ be the versal deformation of a local complex space X . If S is smooth at its origin (or more generally, *rigid*) then V is rigid. This is proved by considering the exact sequence of cotangent spaces ([9]):

$$\cdots \rightarrow T_S^0 \otimes_{\mathcal{O}_S} \mathcal{O}_V \xrightarrow{a} T_{V/S}^1 \xrightarrow{b} T_V^1 \xrightarrow{c} T_S^1 \otimes_{\mathcal{O}_S} \mathcal{O}_V \rightarrow \cdots$$

where $T_S^0 = \text{Der}(\mathcal{O}_S, \mathcal{O}_S)$ is the tangent sheaf to S , and $T_{V/S}^1 = T^1(V/S, \mathcal{O}_V)$ is the relative cotangent sheaf. It follows directly from the completeness property of the versal deformation (called "smoothness" in [11]) that a is a surjection; on the other hand $T_S^1 = 0$ by assumption, so that $T_V^1 = 0$.

Conversely, every rigid singularity is the total space of its own versal deformation.

For example, it has been shown by Dock Sang Rim and also Mary Schaps (unpublished) that the versal deformation of the three coordinate axes in \mathbb{C}^3 is $\mathbb{C}_{\mathbb{P}^1 \times \mathbb{P}^2} \rightarrow \mathbb{C}^3$, and for the four coordinate axes in \mathbb{C}^4 , it is $V \rightarrow \mathbb{C}_{\mathbb{P}^1 \times \mathbb{P}^3}$ where V is an (as yet) unknown rigid singularity. For a local complete intersection X , with versal deformation $V \rightarrow S$ (S is necessarily smooth), V is smooth (for it must also be a complete intersection).

Remark. I know of no rigid singularities in dimension 0 or 1, nor any *normal* rigid surfaces. In fact there is a long standing conjecture that every (reduced) curve may be deformed to a smooth curve, proved for curves in \mathbb{C}^3 by Mary Schaps [13]. Anthony Iarrobino [8] has exhibited an Artin

local space X (dimension 0) which cannot be deformed into n distinct points; the product of X with a line is then a (non-reduced) curve without smooth deformations.

The above ideas may be relevant to these problems; if X is an Artin local scheme (resp. irreducible curve) which has a one-parameter deformation $V \rightarrow S$ (S a disc in \mathbb{C}) such that the corresponding infinitesimal deformation ϕ in T_X^1 generates T_X^1 as an \mathcal{O}_X -module, then V is a rigid curve (resp. normal surface), as may be deduced from the sequence above.

§4. Deformation of Cones

Let $Y \subset \mathbb{P}^n$ be a projective manifold of positive dimension, $C_Y \subset \mathbb{C}^{n+1}$ the cone over Y . We are going to show that, under suitable conditions, every deformation of C_Y is also a cone, over some deformation of Y in \mathbb{P}^n . We compare the infinitesimal deformations and obstructions of Y (in \mathbb{P}^n) to those of C_Y , and employ Proposition 1 of §2 to conclude that the respective versal parameter spaces are isomorphic.

We assume throughout that Y is projectively normal in Y , in other words we have a surjection

$$(7) \quad H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(v)) \rightarrow H^0(Y, \mathcal{O}_Y(v)) \rightarrow 0$$

for all v . (Here we use as before the standard notation $F(v)$ for the tensor product of the sheaf F with the sheaf of sections of the v th power of the hyperplane line bundle on \mathbb{P}^n .)

Equivalently the cone C_Y is the (normal) affine variety

$$C_Y = \text{Spec} \sum_{v \geq 0} H^0(Y, \mathcal{O}_Y(v)).$$

4.1 Projective deformations

Let Y be a projective manifold in \mathbb{P}^n . We consider projective deformations of Y . Such a deformation will be a flat map $\pi: W \rightarrow T$ where T is a local complex space, together with a closed immersion $W \hookrightarrow \mathbb{P}^n \times T$, respecting π , such that $\pi^{-1}(0) = Y$. Two deformations $W \rightarrow T$, $W' \rightarrow T$ are *isomorphic* if there is a (relatively linear) automorphism of $\mathbb{P}^n \times T$ over T taking W to W' . Then $Y \subset \mathbb{P}^n$ has a versal projective deformation $V \rightarrow S$ (the definition is exactly analogous to that for local complex spaces). In fact, one has a formal versal deformation $\tilde{V} \rightarrow \tilde{S}$, where \tilde{S} is spec of a complete local ring by [11], and the Artin approximation theorem [1] proves that this deformation is algebraizable so that $\tilde{S} = S$, where S is algebraic.

As is well known, an infinitesimal variation Y of \mathbb{P}^n is uniquely given by a section of $H^0(Y, N_Y)$, where N_Y is the normal bundle of Y in \mathbb{P}^n . To get t_s from this we must divide out by the space of infinitesimal projective au-

tomorphisms. But a tangent vector field of \mathbf{P}^n is just a homogeneous linear transformation, i.e., an element of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))^{n+1}$. By our projective normality assumption (7) the restriction of these to Y is $H^0(Y, \mathcal{O}_Y(1))^{n+1}$. Thus we obtain the presentation

$$(8) \quad H^0(Y, \mathcal{O}_Y(1))^{n+1} \rightarrow H^0(Y, N_Y) \rightarrow t_S \rightarrow 0.$$

The obstructions to varying Y in \mathbf{P}^n lie in $H^1(Y, N_Y)$, so we have an inclusion

$$(9) \quad \mathcal{O}_S \hookrightarrow H^1(Y, N_Y).$$

4.2 Deformations of C_Y

Let $p: L \rightarrow Y$ be the anti-hyperplane bundle over Y , which contains Y as zero section. Then C_Y is the Remmert quotient of L , in which Y is collapsed to the vertex of C_Y . Letting U be the complement of the vertex in C_Y , we have $U \cong L - Y$ and we can apply Theorem 1 of §2 via the following lemma.

Lemma 1. *Let $p: L \rightarrow Y$ be an algebraic line bundle over the algebraic variety Y , $U = L - Y$.*

Then there is an exact sequence

$$(10) \quad 0 \rightarrow \mathcal{O}_U \rightarrow \Theta_U \rightarrow p^* \Theta_Y \rightarrow 0$$

and for every coherent sheaf F on Y we have canonical isomorphisms

$$(11) \quad H^q(U, p^*F) \cong \sum_{v=-\infty}^{\infty} H^q(Y, F(L^v)).$$

Proof. The bundle of relative tangents of L over Y is p^*L , and so we have

$$0 \rightarrow p^*L \rightarrow \Theta_L \rightarrow p^* \Theta_Y \rightarrow 0.$$

Clearly $p^*L|_U$ is trivial, so restricting to U we get (10).

As an algebraic variety L is $\text{Spec } \sum_{v \geq 0} \mathcal{O}(L^v)$, and U is $\text{Spec } \sum_{-\infty < v < \infty} \mathcal{O}(L^v)$. Thus $p_*(p^*F|_U) = \sum_{-\infty < v < \infty} F(L^v)$, and since $p: U \rightarrow Y$ has Stein fibers (punctured affine lines), $H^q(U, p^*F) \cong H^q(Y, p_*p^*F)$ and (11) follows.

By Serre's Normality Criterion, or by an immediate application of (11) for $F = \mathcal{O}_Y$, $q = 0$, we know that C_Y has depth ≥ 2 at its vertex. We apply Theorem 1 of §2, noting that the normal bundle N_U of U in C^{n+1} is p^*N_Y , and $\mathcal{O}_{C^{n+1}}$ is a free sheaf whose restriction to U is

$$\mathcal{O}_{C^{n+1}}|_U \cong (p^*\mathcal{O}_Y(1))^{n+1}.$$

For the versal parameter space S' of C_Y we have therefore

$$(8') \quad \sum H^0(Y, \mathcal{O}_Y(v+1))^{n+1} \rightarrow \sum H^0(Y, N_Y(v)) \rightarrow t_{S'} \rightarrow 0$$

$$(8'') \quad 0 \rightarrow t_{S'} \rightarrow H^1(U, \Theta_U) \rightarrow \sum H^1(Y, \mathcal{O}_Y(v+1))^{n+1}$$

$$(9') \quad \mathcal{O}_{S'} \curvearrowright \sum H^1(Y, N_Y(v))$$

where all summations are extended over $-\infty < v < \infty$.

4.3 Comparison of S with S'

Suppose again that Y is a projectively normal manifold in \mathbf{P}^n , of positive dimension. Given a projective deformation $W \rightarrow T$ of Y in \mathbf{P}^n , $W \subset \mathbf{P}^n \times T$, we consider the ring

$$A = \sum_{v \geq 0} H^0(W, \mathcal{O}_W(v)).$$

From (7) we see that $H^0(W, \mathcal{O}_W(v)) \rightarrow H^0(Y, \mathcal{O}_Y(v))$ is surjective, so that by [6] (§7.8) A is flat over \mathcal{O}_T . Thus

$$C_W = \text{Spec} \sum_{v \geq 0} H^0(W, \mathcal{O}_W(v))$$

is a deformation of C_Y over T , and we can associate to every projective deformation of Y a deformation of C_Y . It is easy to see that two projective deformations are isomorphic if and only if the corresponding deformations of C_Y are isomorphic, but it may well happen that C_Y has non-conical deformations. (For example when Y is a hypersurface $f=0$, this is certainly so; C_Y even has smooth deformations $f+t=0$.)

Let $V \rightarrow S$ (resp. $V' \rightarrow S'$) be the versal deformation of Y in \mathbf{P}^n (resp. of C_Y). We have an induced map

$$\phi: S \rightarrow S'$$

such that $\phi^*(V') \cong C_Y$.

From (8) and (8') it follows that t_S is included in $t_{S'}$, as the degree 0 component, and from (9), (9') we see that \mathcal{O}_S is the degree 0 component of $\mathcal{O}_{S'}$. By Proposition 1, §2, we conclude

Lemma 2. *If the graded vector space $T_{C_Y}^1 \cong t_{S'}$ is concentrated in degree 0, then $S \cong S'$ and every deformation of C_Y is conical. Thus*

Theorem 2. *Let $Y \subset \mathbf{P}^n$ be a projective manifold of dimension ≥ 2 . Suppose*

- (i) Y is projectively normal
- (ii) $H^1(Y, \mathcal{O}_Y(v)) = 0, \quad v \neq 0$
- (iii) $H^1(Y, \Theta_Y(v)) = 0, \quad v \neq 0.$

Then the versal deformation spaces of Y in \mathbf{P}^n and C_Y are isomorphic; every deformation of C_Y is conical.

Notice that every sufficiently ample projective embedding of Y will have the properties (i), (ii), (iii).

Proof of Theorem 2. From (8''), it suffices to show that $H^1(U, \Theta_U)$ is concentrated in degree 0. From Lemma 1 (10) and (11) applied to $F = \mathcal{O}_Y$ and Θ_Y , we obtain Theorem 2.

This theorem may be rephrased by stating that the correspondence

$$W \mapsto C_W$$

described above is an equivalence between the deformation theories (fibered categories) of Y in \mathbf{P}^n and C_Y .

§5. Generic Singularities Which Are Not Rigid

Let $Y \subset \mathbf{P}^n$ be a projectively normal manifold. Then the cotangent spaces $T_{C_Y}^i$ are graded:

$$T_C^i = \sum (T_C^i)_m, \quad -\infty < m < \infty,$$

and we have seen above that the degree 0 part of T_C^i is T_Y^i , the corresponding space for projective deformations of Y in \mathbf{P}^n .

The cone C^v over the v th Veronese embedding $Y^v \subset \mathbf{P}^{nv}$ of Y is the quotient space of C under the action

$$x_j \mapsto e^{2\pi i j/v} \cdot x_j, \quad 0 \leq j \leq n.$$

Thus if C_Y has depth ≥ 3 at its vertex (equivalently, $H^1(Y, \mathcal{O}_Y(m)) = 0$, all m) then $T_{C^v}^i$ is the invariant subspace of T_C^i under this action, so that

$$T_{C^v}^i \cong \sum_{v|m} (T_C^i)_m.$$

In particular, the part of degree 0 is the same: $T_Y^i \cong T_{Y^v}^i$. This means that Y and Y^v have the same versal parameter space, and in fact, the versal projective deformations for Y and Y^v are respectively:

$$\begin{array}{ccc} W \subset \mathbf{P}^n \times S & & W^v \subset \mathbf{P}^{nv} \times S \\ \downarrow & \text{and} & \downarrow \\ S & & S \end{array}$$

For v sufficiently large, $T_{C^v}^1$ is concentrated in degree 0, and $C_{W^v} \rightarrow S$ is the versal deformation of C^v .

Example. Let $Y: f(x_0, \dots, x_n) = 0$ be a smooth hypersurface of degree d and dimension $n - 1 \geq 2$ in \mathbf{P}^n . C_Y has depth n at its vertex and T_C^1 is the graded vector space given by

$$T_C^1(-d) = \mathbf{C}[x_0, \dots, x_n] / \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n} \right).$$

In other words, a homogeneous polynomial of degree m in the right hand side has degree $m - d$ in T_C^1 . (Notice that f is included in the ideal generated by the $\partial f / \partial x_i$.)

If g_1, \dots, g_r are homogeneous polynomials of degree d which induce a basis modulo $(\partial f / \partial x_0, \dots, \partial f / \partial x_n)$, Y has versal projective deformation

$$W: f(x) + \sum t_i g_i(x) = 0$$

over $S = \mathbf{C}^r$.

The dimension of $(T_C^1)_m$ is independent of the particular smooth hypersurface of degree d chosen, and vanishes when

$$m < -d \text{ or } m > nd - (n + 1).$$

Then for $v > (d, nd - (n + 1))$ the cone $X = C_{Y^v}$ has versal deformation

$$\pi: V = C_{W^v} \rightarrow S$$

in which none of the fibers $X_t = \pi^{-1}(t)$ is rigid. In fact the dimension of $T_{X_t}^1$ is independent of t , and π is versal at each $t \in S$.

The singularity X has, however, a certain "generic" character: it is not the specialization of any other singularity X' . Here, we give the

Definition. Let X and X' be Stein spaces. We say that X' specializes to X if there is a flat map

$$\rho: W \rightarrow T$$

with T a disc: $|t| < a$ in \mathbf{C} , such that within a neighborhood of the singular locus of ρ we have

$$\rho^{-1}(0) \cong X, \quad \rho^{-1}(t) \cong X', \quad t \neq 0.$$

Since $\pi: V \rightarrow S$ is versal at each t in S , S has no positive dimensional subspaces along which the isomorphism class of $\pi^{-1}(t)$ is constant. Therefore X , and also each of the other fibers $X_t = \pi^{-1}(t)$, is not a specialization of any other singularity—and is thus "generic."

REFERENCES

- [1] ARTIN, M., "Algebraization of Formal Moduli, I," *Global Analysis* (papers in honor of K. Kodaira) (University of Tokyo Press, Princeton University Press, 1969).
- [2] BOURBAKI, N., *Algèbre Commutative*, Chap. III (Actualités Sci. Ind., 1923).
- [3] BRIESKORN, R., "Rationale Singularitäten komplexen Flächen," *Inventiones Math.* **4** (1968), 336–358.
- [4] GRAUERT, H., "Über die Deformationen isolierter Singularitäten analytischer Mengen," *Inventiones Math.*, **15** No. 3 (1972).
- [5] ———, and KERNER, H. "Deformationen von Singularitäten komplexer Räume," *Math. Annalen* **153** (1964), 236–260.
- [6] GROTHENDIECK, A., "Éléments de Géométrie Algébrique, III," *Publ. Math. I.H.E.S.*, No. 17.
- [7] HARTSHORNE, R., "Local Cohomology," *Lecture Notes* No. 41 (Berlin-Heidelberg-New York: Springer-Verlag, 1967).
- [8] IARROBINO, A., "Reducibility of the Families of 0-Dimensional Subschemes on a Variety," *Inventiones Math.* **15** (1972), 72–77.
- [9] LICHTENBAUM, S., and M. SCHLESSINGER, "On the Cotangent Complex of a Morphism," *Trans. A.M.S.* **128** (1967), 41–70.
- [10] RIM, D. S., "Torsion Differentials and Deformation," *Trans. A.M.S.* **169** (1972), 257–278.
- [11] SCHLESSINGER, M., "Functors of Artin Rings," *Trans. A.M.S.* **130** (1968), 208–222.
- [12] ———, "Rigidity of Quotient Singularities," *Inventiones Math.* **14** (1971), 17–26.
- [13] SCHAPS, M., "Nonsingular Deformations of Space Curves Using Determinantal Schemes," Harvard thesis (1972).
- [14] TIJOURINA, G. N., "Locally Flat Deformations of Isolated Singularities of Complex Spaces," *Math. of the USSR- Izvestia*, Vol. 3, No. 5 (1969), 967–999.